Threshold Functions in Random s-Intersection Graphs

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***Abstract*—Random s-intersection graphs have recently received considerable attention in a wide range of application areas. In such a graph, each vertex is equipped with a set of items in some random manner, and any two vertices establish an undirected edge in between if and only if they have at least s common items. In particular, in a *uniform* random s-intersection graph, each vertex *independently* selects a fixed number of items uniformly at random from a common item pool, while in a *binomial* random s-intersection graph, each item in some item pool is *independently* attached to each vertex with the same probability.**

For binomial/uniform random s-intersection graphs, we establish threshold functions for perfect matching containment, Hamilton cycle containment, and k-robustness, where k-robustness is in the sense of Zhang and Sundaram [23]. We show that these threshold functions resemble those of classical Erdős–Rényi graphs, where each pair of vertices has an undirected edge independently with the same probability.

***Index Terms*—Hamilton cycle, perfect matching, robustness, threshold function, random s-intersection graph.**

I. Introduction

Random s-intersection graphs have received much interest recently [1]–[5], [7], [9], [15], [18]–[20], [22], [25], [26]. In such a graph, each vertex is equipped with a set of items in some *random* manner, and two vertices establish an undirected edge in between if and only if they share at least s items. Random s-intersection graphs have been used in various applications including secure sensor networks [3], [22], [26], social networks [7], [25], clustering [7], and cryptanalysis [2].

Among different models of random s-intersection graphs, two widely studied models are the so-called *uniform random s-intersection graph* and *binomial random s-intersection graph* [3], [25], which are defined in detail below.

A *binomial s-intersection graph* denoted by $G_s(n, t_n, P_n)$ is defined on n vertices as follows [3], [25]. Each item from a pool of P_n distinct items is assigned to each vertex *independently* with probability t_n. Two vertices establish an undirected edge in between if and only if they have no less than s items in common. The word "binomial" is used since the number of items on each vertex follows a binomial distribution with parameters P_n (the number of trials) and t_n (the success probability in each trial). t_n and P_n are both functions of n, while s does not scale with n. Also it holds that $1 \leq s \leq P_n$.

A *uniform s-intersection graph* denoted by $H_s(n, K_n, P_n)$ is defined on n vertices as follows [3], [25]. Each vertex *independently* selects K_n different items *uniformly at random* from a pool of P_n distinct items. Two vertices have an undirected edge in between if and only if they have at least s common items. The notion "uniform" means that all vertices have the same number of items (but likely different sets of items). K_n and P_n are both functions of n, while s does not scale with n. It holds that $1 \leq s \leq K_n \leq P_n$.

An important application of uniform s-intersection graphs is to model the topologies of secure wireless sensor networks employing the Chan–Perrig–Song key predistribution scheme [8], which is widely recognized as an appropriate solution to secure communications between sensors. In the Chan–Perrig–Song key predistribution scheme for an n-size sensor network, prior to deployment, each sensor is assigned a set of K_n distinct cryptographic keys selected uniformly at random from the same key pool containing P_n different keys. After deployment, two sensors establish secure communication if and only if they have at least s common key(s). Clearly the induced topology is a uniform s-intersection graph.

Our main goal in this paper is to derive the threshold functions of uniform s-intersection graphs and binomial s-intersection graphs for properties including perfect matching containment, Hamilton cycle containment, and k-robustness. These properties are defined as follows: (i) A perfect matching is a set of edges that do not have common vertices and cover all vertices with the exception of missing at most one vertex. (ii) A Hamiltonian cycle means a closed loop that visits each vertex exactly once. (iii) The notion of k-robustness proposed by Zhang and Sundaram [23] measures the effectiveness of local-information-based diffusion algorithms in the presence of adversarial vertices; formally, a graph with a vertex set $\mathcal{V}$ is k-robust if at least one of (a) and (b) below holds for each non-empty and strict subset T of $\mathcal{V}$: (a) there exists at least a vertex $v_a \in T$ such that v_a has no less than k neighbors inside $\mathcal{V} \setminus T$, and (b) there exists at least a vertex $v_b \in \mathcal{V} \setminus T$ such that v_b has no less than k neighbors inside T, where two vertices are neighbors if they have an edge in between.

The above studied properties of uniform s-intersection graphs and binomial s-intersection graphs have diverse applications. First, in the use of uniform s-intersection graphs for secure wireless sensor networks [3], [8], perfect matchings have been used for the optimal allocation of rate and power [17], the design of routing schemes supporting data fusion [12], and the dispatch of sensors [21] (i.e., moving sensors to areas of interest), while Hamilton cycles have been used for cyclic routing which with distributed optimization achieves efficient in-network data processing [16]. Second, in the application of binomial s-intersection graphs to classification and clustering [6], perfect matchings have been used to analyze linear inverse problems [14], while Hamilton cycles have been used to study probabilistic graphical models [13]. Third, the property of k-robustness plays a key role in many classes of dynamics in graphs, such as resilient consensus, contagion and bootstrap percolation [23].

We obtain threshold functions of binomial s-intersection graphs and uniform s-intersection graphs for perfect matching

containment, Hamilton cycle containment, and k-robustness, and show that these thresholds resemble those of Erdős–Rényi graphs [10], where an Erdős–Rényi graph is constructed by assigning an edge between each pair of vertices independently with the same probability. Specifically, just like Erdős–Rényi graphs, for both binomial s-intersection graphs and uniform s-intersection graphs, the thresholds of the edge probability (i.e., the probability of an edge existence between two vertices) are given by

- $\frac{\ln n}{n}$ for perfect matching containment,
- $\frac{\ln n+\ln\ln n}{n}$ for Hamilton cycle containment, and
- $\frac{\ln n+(k-1)\ln\ln n}{n}$ for k-robustness.

We organize the rest of the paper as follows. In Section II, we present the results as theorems, which are proved in Section III. We discuss related work in Section IV and conclude the paper in Section V. The Appendix provides useful lemmas and their proofs.

II. Results

In Sections II-A and II-B below, we summarize our results of binomial random s-intersection graphs and uniform random s-intersection graphs, respectively. Afterwards, we discuss the threshold functions in Section II-C.

Notation and convention: We denote the edge probability of a binomial random s-intersection graph $G_s(n,t_n,P_n)$ by b_n, and denote the edge probability of a binomial random s-intersection graph $H_s(n,K_n,P_n)$ by u_n. Both k and s are constants and do not scale with n. All asymptotic statements are understood with $\to\infty$. We use the Landau asymptotic notation $O(\cdot),o(\cdot),\Omega(\cdot),\omega(\cdot),\Theta(\cdot),\sim$; in particular, for two positive sequences x_n and y_n, the relation $x_n\sim y_n$ signifies $\lim_{n\to\infty}(x_n/y_n)=1$. Also, $\mathbb{P}[\mathcal{E}]$ denotes the probability that event $\mathcal{E}$ occurs. An event happens *asymptotically almost surely* if its probability converges to 1 as $n\to\infty$.

A. Results of binomial random s-intersection graphs

We present results of a binomial random s-intersection graph $G_s(n,t_n,P_n)$ in Theorems 1–3 below. The conditions can be either about the edge probability b_n or its asymptotics $\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^s$ (our work [25, Lemma 12] proves $b_n\sim\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^s$ under certain conditions).

Theorem 1 (Perfect matching containment in binomial random s-intersection graphs). *For a binomial random s-intersection graph $G_s(n,t_n,P_n)$ under $P_n=\Omega(n^c)$ for some constant $c>2-\frac{1}{s}$, under **either** of the following two conditions for all n with a sequence α_n satisfying $\lim_{n\to\infty}\alpha_n=\alpha^*\in[-\infty,\infty]$:*

(i) the edge probability b_n equals $\frac{\ln n+\alpha_n}{n}$,

(ii) $\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^s=\frac{\ln n+\alpha_n}{n}$,

then

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n,P_n)\text{ contains a perfect matching.}]=e^{-e^{-\alpha^*}},$$

which implies that $G_s(n,t_n,P_n)$ asymptotically almost surely does not have a perfect matching if $\alpha^*=-\infty$, and asymptotically almost surely has a perfect matching if $\alpha^*=\infty$.

Theorem 2 (Hamilton cycle containment in binomial random s-intersection graphs). *For a binomial random s-intersection graph $G_s(n,t_n,P_n)$ under $P_n=\Omega(n^c)$ for some constant $c>2-\frac{1}{s}$, under **either** of the following two conditions for all n with a sequence β_n satisfying $\lim_{n\to\infty}\beta_n=\beta^*\in[-\infty,\infty]$:*

(i) the edge probability b_n equals $\frac{\ln n+\ln\ln n+\beta_n}{n}$,

(ii) $\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^s=\frac{\ln n+\ln\ln n+\beta_n}{n}$,

then

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n,P_n)\text{ contains a Hamilton cycle.}]=e^{-e^{-\beta^*}},$$

which implies that $G_s(n,t_n,P_n)$ asymptotically almost surely does not have a Hamilton cycle if $\beta^*=-\infty$, and asymptotically almost surely has a Hamilton cycle if $\beta^*=\infty$.

Theorem 3 (k-Robustness in binomial random s-intersection graphs). *For a binomial random s-intersection graph $G_s(n,t_n,P_n)$ under $P_n=\Omega(n^c)$ for some constant $c>2-\frac{1}{s}$, under **either** of the following two conditions for all n with a sequence γ_n satisfying $\lim_{n\to\infty}\gamma_n=\gamma^*\in[-\infty,\infty]$:*

(i) the edge probability b_n equals $\frac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}$,

(ii) $\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^s=\frac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}$,

then

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n,P_n)\text{ is }k\text{-robust.}]=\begin{cases}0, & \text{if }\gamma^*=-\infty, & \text{(1a)}\\ 1, & \text{if }\gamma^*=\infty. & \text{(1b)}\end{cases}$$

B. Results of uniform random s-intersection graphs

We present results of a uniform random s-intersection graph $H_s(n,K_n,P_n)$ in Theorems 4–6 below. The conditions can be either about the edge probability u_n or its asymptotics $\frac{1}{s!}\cdot\frac{{K_n}^{2s}}{{P_n}^s}$ (our work [25, Lemma 8] shows $u_n\sim\frac{1}{s!}\cdot\frac{{K_n}^{2s}}{{P_n}^s}$ under certain conditions).

Theorem 4 (Perfect matching containment in uniform random s-intersection graphs). *For a uniform random s-intersection graph $H_s(n,K_n,P_n)$ under $P_n=\Omega(n^c)$ for some constant $c>2-\frac{1}{s}$, under **either** of the following two conditions for all n with a sequence α_n satisfying $\lim_{n\to\infty}\alpha_n=\alpha^*\in[-\infty,\infty]$:*

(i) the edge probability u_n equals $\frac{\ln n+\alpha_n}{n}$,

(ii) $\frac{1}{s!}\cdot\frac{{K_n}^{2s}}{{P_n}^s}=\frac{\ln n+\alpha_n}{n}$,

then

$$\lim_{n\to\infty}\mathbb{P}[H_s(n,K_n,P_n)\text{ contains a perfect matching.}]=e^{-e^{-\alpha^*}},\tag{2}$$

which implies that $H_s(n,K_n,P_n)$ asymptotically almost surely does not have a perfect matching if $\alpha^*=-\infty$, and asymptotically almost surely has a perfect matching if $\alpha^*=\infty$.

Theorem 5 (Hamilton cycle containment in uniform random s-intersection graphs). *For a uniform random s-intersection graph $H_s(n,K_n,P_n)$ under $P_n=\Omega(n^c)$ for some constant $c>2-\frac{1}{s}$, under **either** of the following two conditions for all n with a sequence β_n satisfying $\lim_{n\to\infty}\beta_n=\beta^*\in[-\infty,\infty]$:*

(i) the edge probability u_n equals $\frac{\ln n+\ln\ln n+\beta_n}{n}$,

(ii) $\frac{1}{s!}\cdot\frac{{K_n}^{2s}}{{P_n}^s}=\frac{\ln n+\ln\ln n+\beta_n}{n}$,

then

$$\lim_{n\to\infty}\mathbb{P}[H_s(n,K_n,P_n)\text{ contains a Hamilton cycle.}]=e^{-e^{-\beta^*}},\tag{3}$$

which implies that $H_s(n, K_n, P_n)$ asymptotically almost surely does not have a Hamilton cycle if $\beta^* = -\infty$, and asymptotically almost surely has a Hamilton cycle if $\beta^* = \infty$.

Theorem 6 (**k-Robustness in uniform random s-intersection graphs**). *For a uniform random s-intersection graph $H_s(n, K_n, P_n)$ under $P_n = \Omega(n^c)$ for some constant $c > 2 - \frac{1}{s}$, under* ***either*** *of the following two conditions for all n with a sequence γ_n satisfying $\lim_{n\to\infty} \gamma_n = \gamma^* \in [-\infty, \infty]$:*

(i) the edge probability u_n equals $\frac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}$,

(ii) $\frac{1}{s!} \cdot \frac{{K_n}^{2s}}{{P_n}^s} = \frac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}$,

then

$$\lim_{n\to\infty} \mathbb{P}[H_s(n, K_n, P_n) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \gamma^* = -\infty, \\ 1, & \text{if } \gamma^* = \infty. \end{cases} \tag{4}$$

C. Threshold functions in random s-intersection graphs

From Theorems 1–6 above and Appendix-B on Erdős–Rényi graphs, we obtain that the threshold functions of binomial s-intersection graphs and uniform s-intersection graphs for the three studied properties have the same form as those of Erdős–Rényi graphs. Specifically, for a binomial s-intersection graph, a uniform s-intersection graph, and an Erdős–Rényi graph, the thresholds of the edge probability are $\frac{\ln n}{n}$ for perfect matching containment, $\frac{\ln n+\ln\ln n}{n}$ for Hamilton cycle containment, and $\frac{\ln n+(k-1)\ln\ln n}{n}$ for k-robustness.

III. Establishing Theorems 1–6

We use PM and HC and to stand for perfect matching and Hamilton cycle, respectively.

A. Proof of Theorem 1

Theorem 1 follows once we prove

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a PM.}] \le e^{-e^{-\alpha^*}} \cdot [1 + o(1)] \tag{5}$$

and

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a PM.}] \ge e^{-e^{-\alpha^*}} \cdot [1 - o(1)]. \tag{6}$$

(5) clearly holds from Lemma 5 in Appendix-B with $k = 1$ and the fact [19] that a necessary condition for a graph to contain a PM is that the minimum degree is at least 1 (i.e., there is no isolated vertex).

Now we establish (6). From Lemmas 1 and 2 in Appendix-A and the fact that PM containment is a monotone increasing graph property, we can introduce an auxiliary condition $|\alpha_n| = O(\ln\ln n)$. Then we explain that under $|\alpha_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 1 yields

$$\left|\tfrac{1}{s!} \cdot {t_n}^{2s}{P_n}^s - \tfrac{\ln n+\alpha_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{7}$$

Clearly, (7) holds under condition (ii). To show (7) under condition (i) with $|\alpha_n| = O(\ln\ln n)$, we use [25, Lemma 12] to derive $\frac{1}{s!} \cdot {t_n}^{2s}{P_n}^s = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+\alpha_n \pm o(1)}{n}$, which implies (7). Therefore, (7) follows, which with $|\alpha_n| = O(\ln\ln n)$ further induces

$$\tfrac{1}{s!} \cdot {t_n}^{2s}{P_n}^s = \tfrac{\ln n}{n} \cdot [1 \pm o(1)]. \tag{8}$$

We now use Lemmas 7 and 11 in the Appendix to prove (6). We show that the conditions of Lemma 11 all hold given (8) and the condition on P_n in Theorem 1: $P_n = \Omega(n^c)$ for some constant $c > 2 - \frac{1}{s}$. We have ${t_n}^2 P_n = \sqrt[s]{s! \cdot \left(\frac{1}{s!} \cdot {t_n}^{2s}{P_n}^s\right)} = \Theta\left(n^{-\frac{1}{s}}(\ln n)^{\frac{1}{s}}\right)$ so that ${t_n}^2 P_n = o\left(\frac{1}{\ln n}\right)$ and ${t_n}^2 P_n = \omega\left(\frac{1}{n^2}\right)$. Also, we obtain

$$t_n = \sqrt[2s]{s!\left(\tfrac{1}{s!}{t_n}^{2s}{P_n}^s\right)/\left({P_n}^s\right)} = O\left((\ln n)^{\frac{1}{2s}} n^{-\frac{1}{2}(c+\frac{1}{s})}\right) = o\left(\tfrac{1}{n}\right)$$

and

$$t_n P_n = \sqrt[2s]{s!\left(\tfrac{1}{s!}{t_n}^{2s}{P_n}^s\right){P_n}^s} = \Omega\left(n^{\frac{cs-1}{2s}}(\ln n)^{\frac{1}{2s}}\right) = \omega(\ln n),$$

where the last step applies $cs > 2s - 1 \ge 1$. Hence, all conditions of Lemma 11 hold. Then from Lemma 10, Lemma 11, and the monotonicity of PM containment, there exists a sequence h_n satisfying

$$h_n = \tfrac{1}{s!} \cdot {t_n}^{2s}{P_n}^s \cdot \left[1 - o\left(\tfrac{1}{\ln n}\right)\right] \tag{9}$$

such that

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a PM.}] \ge \mathbb{P}[G_{ER}(n, h_n) \text{ has a PM.}] - o(1). \tag{10}$$

Substituting (7) and (8) into (9), we derive $h_n = \frac{\ln n+\alpha_n \pm o(1)}{n}$, which is used in Lemma 7 to induce

$$\lim_{n\to\infty} \mathbb{P}[G_{ER}(n, h_n) \text{ has a PM.}] = e^{-e^{-\alpha^*}}. \tag{11}$$

Then (6) clearly follows from (10) and (11).

We have established Theorem 1 by showing (5) and (6).

B. Proof of Theorem 2

Theorem 2 follows once we prove

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a HC.}] \le e^{-e^{-\beta^*}} \cdot [1 + o(1)] \tag{12}$$

and

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a HC.}] \ge e^{-e^{-\beta^*}} \cdot [1 - o(1)]. \tag{13}$$

(12) clearly holds from Lemma 5 with $k = 2$ and the fact [19] that a necessary condition for a graph to contain a HC is that the minimum degree is at least 2.

Now we establish (13). From Lemma 1, Lemma 2 and the fact that HC containment is a monotone increasing graph property, we can introduce an auxiliary condition $|\beta_n| = O(\ln\ln n)$. Then we explain that under $|\beta_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 2 yields

$$\left|\tfrac{1}{s!} \cdot {t_n}^{2s}{P_n}^s - \tfrac{\ln n+\ln\ln n+\beta_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{14}$$

Clearly, (17) holds under condition (ii). To show (17) under condition (i) with $|\beta_n| = O(\ln\ln n)$, we use [25, Lemma 12] to derive $\frac{1}{s!} \cdot {t_n}^{2s}{P_n}^s = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+\ln\ln n+\beta_n \pm o(1)}{n}$, which implies (17). Therefore, (17) follows, which with $|\beta_n| = O(\ln\ln n)$ further induces (8). As explained above in the proof of Theorem 1, all conditions of Lemma 11 hold given (8) and the condition on P_n in Theorem 2: $P_n = \Omega(n^c)$ for some constant $c > 2 - \frac{1}{s}$. Then from Lemma 11, Lemma 10 and the monotonicity of HC containment, there exists a sequence h_n satisfying (9) such that

$$\mathbb{P}[G_s(n, t_n, P_n) \text{ has a HC.}] \ge \mathbb{P}[G_{ER}(n, h_n) \text{ has a HC.}] - o(1). \tag{15}$$

Substituting (17) and (8) into (9), we derive $h_n = \frac{\ln n+\ln\ln n+\beta_n \pm o(1)}{n}$, which is used in Lemma 8 to induce

$$\lim_{n\to\infty} \mathbb{P}[G_{ER}(n, h_n) \text{ has a HC.}] = e^{-e^{-\beta^*}}. \tag{16}$$

Then (13) clearly follows from (18) and (19).

We have established Theorem 2 by showing (12) and (13).

C. Proof of Theorem 3

From [23, Lemma 1], a necessary condition for a graph to be k-robust is that the graph is k-connected, so we clearly obtain (1a) from Lemma 5 in view that

$$\lim_{n\to\infty} \mathbb{P}[G_s(n,t_n,P_n) \text{ is } k\text{-connected.}] = 0 \quad \text{ if } \gamma^* = -\infty,$$

Now we establish (1b). From Lemma 1, Lemma 2 and the fact that HC containment is a monotone increasing graph property, we can introduce an auxiliary condition $|\gamma_n| = O(\ln\ln n)$. Then we explain that under $|\gamma_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 3 yields

$$\left|\tfrac{1}{s!}\cdot {t_n}^{2s}{P_n}^{s} - \tfrac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{17}$$

Clearly, (17) holds under condition (ii). To show (17) under condition (i) with $|\gamma_n| = O(\ln\ln n)$, we use [25, Lemma 12] to derive $\frac{1}{s!}\cdot {t_n}^{2s}{P_n}^{s} = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+(k-1)\ln\ln n+\gamma_n\pm o(1)}{n}$, which implies (17). Therefore, (17) follows, which with $|\gamma_n| = O(\ln\ln n)$ further induces (8). As explained above in the proof of Theorem 1, all conditions of Lemma 11 hold given (8) and the condition on P_n in Theorem 3: $P_n = \Omega(n^c)$ for some constant $c > 2-\frac{1}{s}$. Then from Lemma 11, Lemma 10 and the monotonicity of k-robustness, there exists a sequence h_n satisfying (9) such that

$$\mathbb{P}[G_s(n,t_n,P_n) \text{ is } k\text{-robust.}] \geq \mathbb{P}[G_{ER}(n,h_n) \text{ is } k\text{-robust.}] - o(1). \tag{18}$$

Substituting (17) and (8) into (9), we derive $h_n = \frac{\ln n+(k-1)\ln\ln n+\gamma_n\pm o(1)}{n}$, which is used in Lemma 9 to induce

$$\lim_{n\to\infty} \mathbb{P}[G_{ER}(n,h_n) \text{ is } k\text{-robust.}] = 1 \quad \text{ if } \gamma^* = -\infty. \tag{19}$$

Then (1b) clearly follows from (18) and (19).

We have established Theorem 3 by showing (1a) and (1b).

D. Proof of Theorem 4

From Lemma 3, Lemma 4 and the fact that PM containment is a monotone increasing graph property, we can introduce an auxiliary condition $|\alpha_n| = O(\ln\ln n)$. Then we explain that under $|\alpha_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 4 yields

$$\left|\tfrac{1}{s!}\cdot \tfrac{{K_n}^{2s}}{{P_n}^{s}} - \tfrac{\ln n+\alpha_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{20}$$

Clearly, (20) holds under condition (ii). To show (20) under condition (i) with $|\alpha_n| = O(\ln\ln n)$, we use [25, Lemma 8] to derive $\frac{1}{s!}\cdot \frac{{K_n}^{2s}}{{P_n}^{s}} = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+\alpha_n\pm o(1)}{n}$, which implies (20). Therefore, (20) follows, which with $|\alpha_n| = O(\ln\ln n)$ further induces

$$\tfrac{1}{s!}\cdot \tfrac{{K_n}^{2s}}{{P_n}^{s}} = \tfrac{\ln n}{n}\cdot[1\pm o(1)]. \tag{21}$$

From (21) and $P_n = \Omega(n^c)$ for a constant $c > 2-\frac{1}{s}$, it holds that

$$K_n = \sqrt[2s]{s!\cdot\left(\tfrac{1}{s!}\cdot\tfrac{{K_n}^{2s}}{{P_n}^{s}}\right)\cdot {P_n}^{s}} = \Omega\left(n^{\frac{c}{2}-\frac{1}{2s}}(\ln n)^{\frac{1}{2s}}\right), \tag{22}$$

which clearly implies $K_n = \omega(\ln n)$ so we obtain from Lemma 13, Lemma 10 and the monotonicity of PM containment that

$$\begin{aligned}&\mathbb{P}[G_s(n,t_n^-,P_n) \text{ has a PM.}] - o(1)\\ &\leq \mathbb{P}[H_s(n,K_n,P_n) \text{ has a PM.}]\\ &\leq \mathbb{P}[G_s(n,t_n^+,P_n) \text{ has a PM.}] + o(1),\end{aligned} \tag{23}$$

where

$$t_n^{\pm} = \tfrac{K_n}{P_n}\left(1\pm\sqrt{\tfrac{3\ln n}{K_n}}\right). \tag{24}$$

Then we get from (24) that

$$\tfrac{1}{s!}\cdot\left(t_n^{\pm}\right)^{2s}{P_n}^{s} = \tfrac{1}{s!}\cdot\tfrac{{K_n}^{2s}}{{P_n}^{s}}\cdot\left(1\pm\sqrt{\tfrac{3\ln n}{K_n}}\right)^{2s}. \tag{25}$$

Given (22) and constant s, we have

$$\left(1\pm\sqrt{\tfrac{3\ln n}{K_n}}\right)^{2s} = 1\pm\Theta\left(\sqrt{\tfrac{\ln n}{K_n}}\right) = 1\pm o\left(\tfrac{1}{\ln n}\right), \tag{26}$$

which along with (25) and (20) under $|\alpha_n| = O(\ln\ln n)$ yields

$$\tfrac{1}{s!}\cdot\left(t_n^{\pm}\right)^{2s}{P_n}^{s} = \tfrac{\ln n+\alpha_n\pm o(1)}{n}. \tag{27}$$

Given (27) and $P_n = \Omega(n^c)$ for a constant $c > 2-\frac{1}{s}$, we use Theorem 1 to derive

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n^{\pm},P_n) \text{ has a PM.}] = e^{-e^{-\alpha^*}},$$

which together with (23) induces (2).

E. Proof of Theorem 5

From Lemma 3, Lemma 4 and the fact that HC containment is a monotone increasing graph property, we can introduce an auxiliary condition $|\beta_n| = O(\ln\ln n)$. Then we explain that under $|\beta_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 5 yields

$$\left|\tfrac{1}{s!}\cdot \tfrac{{K_n}^{2s}}{{P_n}^{s}} - \tfrac{\ln n+\ln\ln n+\beta_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{28}$$

Clearly, (28) holds under condition (ii). To show (28) under condition (i) with $|\beta_n| = O(\ln\ln n)$, we use [25, Lemma 8] to derive $\frac{1}{s!}\cdot \frac{{K_n}^{2s}}{{P_n}^{s}} = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+\ln\ln n+\beta_n\pm o(1)}{n}$, which implies (28). Therefore, (28) follows, which with $|\beta_n| = O(\ln\ln n)$ further induces (21). Then (22) holds, and we obtain from Lemma 13, Lemma 10 and the monotonicity of HC containment that

$$\begin{aligned}&\mathbb{P}[G_s(n,t_n^-,P_n) \text{ has a HC.}] - o(1)\\ &\leq \mathbb{P}[H_s(n,K_n,P_n) \text{ has a HC.}]\\ &\leq \mathbb{P}[G_s(n,t_n^+,P_n) \text{ has a HC.}] + o(1),\end{aligned} \tag{29}$$

with t_n^- and t_n^+ specified in (24). Then we also obtain (25) and (26), which together with (20) under $|\beta_n| = O(\ln\ln n)$ lead to

$$\frac{1}{s!}\cdot\left(t_n^{\pm}\right)^{2s}{P_n}^{s} = \frac{\ln n+\ln\ln n+\beta_n\pm o(1)}{n}. \tag{30}$$

Given (30) and $P_n = \Omega(n^c)$ for a constant $c > 2-\frac{1}{s}$, we use Theorem 2 to derive

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n^{\pm},P_n) \text{ has a HC.}] = e^{-e^{-\beta^*}},$$

which along with (29) yields (3).

F. Proof of Theorem 6

From Lemma 3, Lemma 4 and the fact that k-robustness is a monotone increasing graph property, we can introduce an auxiliary condition $|\gamma_n| = O(\ln\ln n)$. Then we explain that under $|\gamma_n| = O(\ln\ln n)$, either of conditions (i) and (ii) in Theorem 6 yields

$$\left|\tfrac{1}{s!}\cdot \tfrac{{K_n}^{2s}}{{P_n}^{s}} - \tfrac{\ln n+(k-1)\ln\ln n+\gamma_n}{n}\right| = o\left(\tfrac{1}{n}\right). \tag{31}$$

Clearly, (31) holds under condition (ii). To show (31) under condition (i) with $|\gamma_n| = O(\ln\ln n)$, we use [25, Lemma 8] to derive $\frac{1}{s!}\cdot \frac{{K_n}^{2s}}{{P_n}^{s}} = u_n \pm o\left(\frac{1}{n}\right) = \frac{\ln n+(k-1)\ln\ln n+\gamma_n\pm o(1)}{n}$, which implies (31). Therefore, (31) follows, which with $|\gamma_n| = O(\ln\ln n)$ further induces (21). Then (22) holds, and we obtain

from Lemma 13, Lemma 10 and the monotonicity of k-robustness that

$$\begin{aligned}&\mathbb{P}[G_s(n,t_n^-,P_n)\text{ is }k\text{-robust.}]-o(1)\\&\leq\mathbb{P}[H_s(n,K_n,P_n)\text{ is }k\text{-robust.}]\\&\leq\mathbb{P}[G_s(n,t_n^+,P_n)\text{ is }k\text{-robust.}]+o(1),\end{aligned}\tag{32}$$

with t_n^- and t_n^+ specified in (24). Then we also obtain (25) and (26), which along with (20) under $|\gamma_n|=O(\ln\ln n)$ result in

$$\frac{1}{s!}\cdot\left(t_n^{\pm}\right)^{2s}{P_n}^s=\frac{\ln n+(k-1)\ln\ln n+\gamma_n\pm o(1)}{n}.\tag{33}$$

Given (33) and $P_n=\Omega(n^c)$ for a constant $c>2-\frac{1}{s}$, we use Theorem 3 to derive

$$\lim_{n\to\infty}\mathbb{P}[G_s(n,t_n^{\pm},P_n)\text{ is }k\text{-robust.}]=\begin{cases}0, & \text{if }\gamma^*=-\infty,\\1, & \text{if }\gamma^*=\infty,\end{cases}$$

which together with (32) yields (4).

IV. Related Work

Binomial s-intersection graphs have been studied as follows. For k-connectivity, we [25] obtain the asymptotically exact probability and specify $\frac{\ln n+(k-1)\ln\ln n}{n}$ as a threshold of the edge probability. Bloznelis *et al.* [3] investigate the component evolution in binomial s-intersection graphs and prove $\frac{1}{n}$ as a threshold of the edge probability for the emergence of a giant component (i.e., a connected subgraph of $\Theta(n)$ vertices).

Uniform s-intersection graphs have also been investigated as follows. For perfect matching containment, Bloznelis and Łuczak [4] give the asymptotically exact probability result, which determines $\frac{\ln n}{n}$ as a threshold of the edge probability, but their result after a rewriting applies to a different set of conditions on P_n compared with our Theorem 1. We require $P_n=\Omega(n^c)$ for a constant $c>2-\frac{1}{s}$, while they consider instead a narrow range of $P_n=\Omega\big(n(\ln n)^{-1}\big)$ and $P_n=o\big(n(\ln n)^{-\frac{3}{5s}}\big)$. For k-connectivity, both our recent paper [25] and another work by Bloznelis and Rybarczyk [5] derive the asymptotically exact probability and determine $\frac{\ln n+(k-1)\ln\ln n}{n}$ as a threshold of the edge probability. However, our result [25] considers $P_n=\Omega(n)$ for $s\geq 2$ or $P_n=\Omega(n^c)$ for $s=1$ with a constant $c>1$, while Bloznelis and Rybarczyk [5] again use $P_n=\Omega\big(n(\ln n)^{-1}\big)$ and $P_n=o\big(n(\ln n)^{-\frac{3}{5s}}\big)$. Bloznelis *et al.* [3] regard the component evolution in uniform s-intersection graphs and show $\frac{1}{n}$ as a threshold of the edge probability for the appearance of a giant component.

A large body of work [1], [9], [15], [18]–[20], [22], [26] study binomial/uniform $\mathbf{1}$-intersection graphs as follows: Rybarczyk [19], [20] investigates k-connectivity, perfect matching containment and Hamilton cycle containment; we [26] consider k-robustness and k-connectivity; Efthymioua and Spirakis [9] and Nikoletseas *et al.* [15] analyze Hamilton cycle containment; and Blackburn and Gerke [1], Rybarczyk [18]–[20], and Yağan and Makowski [22] look at connectivity.

V. Conclusion

In this paper, for binomial/uniform random s-intersection graphs, we establish threshold functions for perfect matching containment, Hamilton cycle containment, and k-robustness. To obtain these results, we derive the asymptotically exact probabilities of perfect matching containment and Hamilton cycle containment, and zero–one laws for k-robustness.

References

[1] S. Blackburn and S. Gerke. Connectivity of the uniform random intersection graph. *Discrete Mathematics*, 309(16), August 2009.

[2] S. Blackburn, D. Stinson, and J. Upadhyay. On the complexity of the herding attack and some related attacks on hash functions. *Designs, Codes and Cryptography*, 64(1-2):171–193, 2012.

[3] M. Bloznelis, J. Jaworski, and K. Rybarczyk. Component evolution in a secure wireless sensor network. *Networks*, 53:19–26, January 2009.

[4] M. Bloznelis and T. Łuczak. Perfect matchings in random intersection graphs. *Acta Mathematica Hungarica*, 138(1-2):15–33, 2013.

[5] M. Bloznelis and K. Rybarczyk. k-connectivity of uniform s-intersection graphs. *Discrete Mathematics*, 333(0):94–100, 2014.

[6] T. Britton, M. Deijfen, A. N. Lagers, and M. Lindholm. Epidemics on random graphs with tunable clustering. *J. Appl. Probab.*, 45(3):743–756, 09 2008.

[7] M. Deijfen and W. Kets, "Random intersection graphs with tunable degree distribution and clustering," *Probability in the Engineering and Informational Sciences*, vol. 23, pp. 661–674, 2009.

[8] H. Chan, A. Perrig, and D. Song. Random key predistribution schemes for sensor networks. In *IEEE Symposium on Security and Privacy*, May 2003.

[9] C. Efthymiou and P. Spirakis. Sharp thresholds for hamiltonicity in random intersection graphs. *Theoretical Computer Science*, 411(40–42):3714–3730, 2010.

[10] P. Erdős and A. Rényi. On the existence of a factor of degree one of a connected random graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 17(3-4):359–368, 1966.

[11] J. Komlós and E. Szemerédi. Limit distribution for the existence of hamiltonian cycles in a random graph. *Discrete Mathematics*, 43(1):55–63, 1983.

[12] H. Luo, Y. Liu, and S. Das. Routing correlated data in wireless sensor networks: A survey. *IEEE Network*, 21(6):40–47, November 2007.

[13] O. Mengshoel, D. Wilkins, and D. Roth. Controlled generation of hard and easy bayesian networks: Impact on maximal clique size in tree clustering. *Artificial Intelligence*, 170(16–17):1137–1174, 2006.

[14] O. Meshi. *Efficient Methods for Learning and Inference in Structured Output Prediction*. PhD thesis, Hebrew University of Jerusalem, August 2013.

[15] S. Nikoletseas, C. Raptopoulos, and P. Spirakis. On the independence number and hamiltonicity of uniform random intersection graphs. *Theoretical Computer Science*, 412(48):6750–6760, 2011.

[16] M. Rabbat and R. Nowak. Quantized incremental algorithms for distributed optimization. *IEEE Journal on Selected Areas in Communications*, 23(4):798–808, April 2005.

[17] A. Roumy and D. Gesbert. Optimal matching in wireless sensor networks. *IEEE Journal of Selected Topics in Signal Processing*, 1(4):725–735, Dec 2007.

[18] K. Rybarczyk. Diameter, connectivity and phase transition of the uniform random intersection graph. *Discrete Mathematics*, 311, 2011.

[19] K. Rybarczyk. Sharp threshold functions for the random intersection graph via a coupling method. *The Electronic Journal of Combinatorics*, 18:36–47, 2011.

[20] K. Rybarczyk. The coupling method for inhomogeneous random intersection graphs. *ArXiv e-prints*, January 2013. Available online at http://arxiv.org/pdf/1301.0466v4.pdf

[21] Y.-C. Wang, C.-C. Hu, and Y.-C. Tseng. Efficient placement and dispatch of sensors in a wireless sensor network. *IEEE Transactions on Mobile Computing*, 7(2):262–274, Feb 2008.

[22] O. Yağan and A. M. Makowski. Zero–one laws for connectivity in random key graphs. *IEEE Transactions on Information Theory*, 58(5):2983–2999, May 2012.

[23] H. Zhang and S. Sundaram. Robustness of complex networks with implications for consensus and contagion. In *IEEE Conference on Decision and Control (CDC)*, pages 3426–3432, December 2012.

[24] J. Zhao, O. Yağan, and V. Gligor. Designing securely and reliably connected wireless sensor networks. *Arxiv e-prints*, 2015. Available online at http://arxiv.org/pdf/1501.01826v1.pdf

[25] J. Zhao, O. Yağan, and V. Gligor. On k-connectivity and minimum vertex degree in random s-intersection graphs. *Arxiv e-prints*, 2014. Available online at http://http://arxiv.org/pdf/1409.6021v3.pdf

[26] J. Zhao, O. Yağan, and V. Gligor. On the strengths of connectivity and robustness in general random intersection graphs. In *IEEE Conference on Decision and Control (CDC)*, December 2014.

APPENDIX

A. Confining α_n in Theorems 1 and 4, β_n in Theorems 2 and 5, and γ_n in Theorems 3 and 6 all as $\pm O(\ln\ln n)$

Lemma 1 (Our work [25, Lemma 2]). *For a binomial random intersection graph $G_s(n, t_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$ and $\frac{1}{s!} \cdot t_n{}^{2s} P_n{}^s = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, the following results hold:*

(i) If $\lim_{n\to\infty} \alpha_n = -\infty$, then there exists graph $G_s(n, \widetilde{t_n}, \widetilde{P_n})$ under $\widetilde{P_n} = \Omega(n^{\widetilde{c}})$ with a constant $\widetilde{c} > 2 - \frac{1}{s}$, and $\frac{1}{s!} \cdot \widetilde{t_n}^{2s} \widetilde{P_n}^s = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\alpha_n}}{n}$ with $\lim_{n\to\infty} \widetilde{\alpha_n} = -\infty$ and $\widetilde{\alpha_n} = -O(\ln\ln n)$, such that $G_s(n, t_n, P_n) \preceq G_s(n, \widetilde{t_n}, \widetilde{P_n})$.

(ii) If $\lim_{n\to\infty} \alpha_n = \infty$, then there exists graph $G_s(n, \widehat{t_n}, \widehat{P_n})$ under $\widehat{P_n} = \Omega(n^{\widehat{c}})$ with a constant $\widehat{c} > 2 - \frac{1}{s}$, and $\frac{1}{s!} \cdot \widehat{t_n}^{2s} \widehat{P_n}^s = \frac{\ln n + (k-1)\ln\ln n + \widehat{\alpha_n}}{n}$ with $\lim_{n\to\infty} \widehat{\alpha_n} = \infty$ and $\widehat{\alpha_n} = O(\ln\ln n)$, such that $G_s(n, \widehat{t_n}, \widehat{P_n}) \preceq G_s(n, t_n, P_n)$.

Lemma 2 (Our work [25, Lemma 16]). *For a binomial random intersection graph $G_s(n, t_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$ and $b_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, where b_n is the edge probability of $G_s(n, t_n, P_n)$, the following results hold:*

(i) If $\lim_{n\to\infty} \alpha_n = -\infty$, then there exists graph $G_s(n, \widetilde{t_n}, \widetilde{P_n})$ under $\widetilde{P_n} = \Omega(n^{\widetilde{c}})$ with a constant $\widetilde{c} > 2 - \frac{1}{s}$, and $\widetilde{b_n} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\alpha_n}}{n}$ with $\lim_{n\to\infty} \widetilde{\alpha_n} = -\infty$ and $\widetilde{\alpha_n} = -O(\ln\ln n)$, where $\widetilde{b_n}$ is the edge probability of $G_s(n, \widetilde{t_n}, \widetilde{P_n})$, such that $G_s(n, t_n, P_n) \preceq G_s(n, \widetilde{t_n}, \widetilde{P_n})$.

(ii) If $\lim_{n\to\infty} \alpha_n = \infty$, then there exists graph $G_s(n, \widehat{t_n}, \widehat{P_n})$ under $\widehat{P_n} = \Omega(n^{\widehat{c}})$ with a constant $\widehat{c} > 2 - \frac{1}{s}$, and $\widehat{b_n} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\alpha_n}}{n}$ with $\lim_{n\to\infty} \widehat{\alpha_n} = \infty$ and $\widehat{\alpha_n} = O(\ln\ln n)$, where $\widehat{b_n}$ is the edge probability of $G_s(n, \widehat{t_n}, \widehat{P_n})$, such that $G_s(n, \widehat{t_n}, \widehat{P_n}) \preceq G_s(n, t_n, P_n)$.

Lemma 3 (Our work [25, Lemma 1]). *For a uniform random s-intersection graph $H_s(n, K_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$ and $\frac{1}{s!} \cdot \frac{K_n{}^{2s}}{P_n{}^s} = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n}$, the following results hold:*

(i) If $\lim_{n\to\infty} \beta_n = -\infty$, then there exists graph $H_s(n, \widetilde{K_n}, \widetilde{P_n})$ under $\widetilde{P_n} = \Omega(n^{\widetilde{c}})$ with a constant $\widetilde{c} > 2 - \frac{1}{s}$, and $\frac{1}{s!} \cdot \frac{\widetilde{K_n}^{2s}}{\widetilde{P_n}^s} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\beta_n}}{n}$ with $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -O(\ln\ln n)$, such that $H_s(n, K_n, P_n) \preceq H_s(n, \widetilde{K_n}, \widetilde{P_n})$.

(ii) If $\lim_{n\to\infty} \beta_n = \infty$, then there exists graph $H_s(n, \widehat{K_n}, \widehat{P_n})$ under $\widehat{P_n} = \Omega(n^{\widehat{c}})$ with a constant $\widehat{c} > 2 - \frac{1}{s}$, and $\frac{1}{s!} \cdot \frac{\widehat{K_n}^{2s}}{\widehat{P_n}^s} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\beta_n}}{n}$ with $\lim_{n\to\infty} \widehat{\beta_n} = \infty$ and $\widehat{\beta_n} = O(\ln\ln n)$, such that $H_s(n, \widehat{K_n}, \widehat{P_n}) \preceq H_s(n, K_n, P_n)$.

Lemma 4 (Our work [25, Lemma 15]). *For a uniform random s-intersection graph $H_s(n, K_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$ and $u_n = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n}$, where u_n is the edge probability of $H_s(n, K_n, P_n)$, the following results hold:*

(i) If $\lim_{n\to\infty} \beta_n = -\infty$, then there exists graph $H_s(n, \widetilde{K_n}, \widetilde{P_n})$ under $\widetilde{P_n} = \Omega(n^{\widetilde{c}})$ with a constant $\widetilde{c} > 2 - \frac{1}{s}$, and $\widetilde{u_n} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\beta_n}}{n}$ with $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -O(\ln\ln n)$, where $\widetilde{u_n}$ is the edge probability of $H_s(n, \widetilde{K_n}, \widetilde{P_n})$, such that $H_s(n, K_n, P_n) \preceq H_s(n, \widetilde{K_n}, \widetilde{P_n})$.

(ii) If $\lim_{n\to\infty} \beta_n = \infty$, then there exists graph $H_s(n, \widehat{K_n}, \widehat{P_n})$ under $\widehat{P_n} = \Omega(n^{\widehat{c}})$ with a constant $\widehat{c} > 2 - \frac{1}{s}$, and $\widehat{u_n} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\beta_n}}{n}$ with $\lim_{n\to\infty} \widehat{\beta_n} = \infty$ and $\widehat{\beta_n} = O(\ln\ln n)$, where $\widehat{u_n}$ is the edge probability of $H_s(n, \widehat{K_n}, \widehat{P_n})$, such that $H_s(n, \widehat{K_n}, \widehat{P_n}) \preceq H_s(n, K_n, P_n)$.

B. Our previous work on random s-intersection graphs for k-connectivity and the property of minimum degree being at least k

Lemma 5 (Our work [25, Theorem 2 and Lemma 14]). *For a binomial random s-intersection graph $G_s(n, t_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$, under **either** of the following two conditions for all n with a sequence δ_n with $\lim_{n\to\infty} \delta_n = \delta^* \in [-\infty, \infty]$:*

(i) the edge probability b_n equals $\frac{\ln n + (k-1)\ln\ln n + \delta_n}{n}$,

(ii) $\frac{1}{s!} \cdot t_n{}^{2s} P_n{}^s = \frac{\ln n + (k-1)\ln\ln n + \delta_n}{n}$,

then

$$\lim_{n\to\infty} \mathbb{P}[G_s(n, t_n, P_n) \text{ is } k\text{-connected.}]$$
$$= \lim_{n\to\infty} \mathbb{P}[G_s(n, t_n, P_n) \text{ has a minmimum degree at least } k.]$$
$$= e^{-\frac{e^{-\delta^*}}{(k-1)!}}.$$

Lemma 6 (Our work [25, Theorem 1 and Lemma 13]). *For a uniform random s-intersection graph $H_s(n, K_n, P_n)$ under $P_n = \Omega(n^c)$ for a constant $c > 2 - \frac{1}{s}$, under **either** of the following two conditions for all n with a sequence δ_n with $\lim_{n\to\infty} \delta_n = \delta^* \in [-\infty, \infty]$:*

(i) the edge probability u_n equals $\frac{\ln n + (k-1)\ln\ln n + \delta_n}{n}$,

(ii) $\frac{1}{s!} \cdot \frac{K_n{}^{2s}}{P_n{}^s} = \frac{\ln n + (k-1)\ln\ln n + \delta_n}{n}$,

then

$$\lim_{n\to\infty} \mathbb{P}[H_s(n, K_n, P_n) \text{ is } k\text{-connected.}]$$
$$= \lim_{n\to\infty} \mathbb{P}[H_s(n, K_n, P_n) \text{ has a minmimum degree at least } k.]$$
$$= e^{-\frac{e^{-\delta^*}}{(k-1)!}}.$$

C. Prior work on Erdős–Rényi graphs for perfect matching containment, Hamilton cycle containment and k-robustness

Lemma 7 ([10, Theorem 1]). *For an Erdős–Rényi graph $G_{ER}(n, h_n)$, if there is a sequence α_n with $\lim_{n\to\infty} \alpha_n \in [-\infty, \infty]$ such that $h_n = \frac{\ln n + \alpha_n}{n}$, then it holds that*

$$\lim_{n\to\infty} \mathbb{P}[G_{ER}(n, h_n) \text{ has a perfect matching.}] = e^{-e^{-\lim_{n\to\infty} \alpha_n}}.$$

Lemma 8 ([11, Theorem 1]). *For an Erdős–Rényi graph $G_{ER}(n, h_n)$, if there is a sequence β_n with $\lim_{n\to\infty} \beta_n \in [-\infty, \infty]$ such that $h_n = \frac{\ln n + \ln\ln n + \beta_n}{n}$, then it holds that*

$$\lim_{n\to\infty} \mathbb{P}[G_{ER}(n, h_n) \text{ has a Hamilton cycle.}] = e^{-e^{-\lim_{n\to\infty} \beta_n}}.$$

Lemma 9 (Our work [26, Lemma 1] based on [23, Theorem 3]). *For an Erdős–Rényi graph $G(n, h_n)$, with a sequence γ_n for all n through*

$$h_n = \frac{\ln n + (k-1)\ln\ln n + \gamma_n}{n}, \tag{34}$$

then it holds that

$$\lim_{n\to\infty} \mathbb{P}\big[G(n,h_n) \text{ is } k\text{-robust.}\big] = \begin{cases} 0, & \text{if } \lim_{n\to\infty}\gamma_n = -\infty, \\ 1, & \text{if } \lim_{n\to\infty}\gamma_n = \infty. \end{cases} \tag{35}$$

D. A coupling between random graphs

Intuitively, a coupling between random graphs is used so that results on the probability of one graph having certain monotone property can help obtain the result on the probability of another graph having the same property [19], [20], [26]. As explained by Rybarczyk [19], [20], a coupling of two random graphs G_1 and G_2 means a probability space on which random graphs G_1' and G_2' are defined such that G_1' and G_2' have the same distributions as G_1 and G_2, respectively. If G_1' is a spanning subgraph (resp., spanning supergraph) G_2', we say that under the graph coupling, G_1 is a spanning subgraph (resp., spanning supergraph) G_2.

Following Rybarczyk's notation [19], we write

$$G_1 \preceq G_2 \quad (\text{resp.}, G_1 \preceq_{1-o(1)} G_2) \tag{36}$$

if there exists a coupling under which G_1 is a spanning subgraph of G_2 with probability 1 (resp., $1-o(1)$).

For two random graphs G_1 and G_2, with $\mathcal{I}$ being a monotone increasing graph property, the following lemma relates $\mathbb{P}\big[G_1 \text{ has } \mathcal{I}.\big]$ and $\mathbb{P}\big[G_2 \text{ has } \mathcal{I}.\big]$.

the probability that

Lemma 10 (Rybarczyk [19])**.** *For two random graphs G_1 and G_2, the following results hold for any monotone increasing graph property $\mathcal{I}$.*

(i) If $G_1 \preceq G_2$, then

$$\mathbb{P}\big[G_2 \text{ has } \mathcal{I}.\big] \geq \mathbb{P}\big[G_1 \text{ has } \mathcal{I}.\big]. \tag{37}$$

(ii) If $G_1 \preceq_{1-o(1)} G_2$, then

$$\mathbb{P}\big[G_2 \text{ has } \mathcal{I}.\big] \geq \mathbb{P}\big[G_1 \text{ has } \mathcal{I}.\big] - o(1). \tag{38}$$

E. Containment of Erdős–Rényi graphs in binomial random intersection graphs

Lemma 11 (Our work [24, Lemma 5])**.** *If ${t_n}^2 P_n = o\big(\frac{1}{\ln n}\big)$, ${t_n}^2 P_n = \omega\big(\frac{1}{n^2}\big)$, $t_n = o\big(\frac{1}{n}\big)$, and $t_n P_n = \omega(\ln n)$, then there exits some h_n satisfying*

$$h_n = \frac{1}{s!} \cdot {t_n}^{2s} {P_n}^{s} \cdot \left[1 - o\left(\frac{1}{\ln n}\right)\right] \tag{39}$$

such that an Erdős–Rényi graph $G_{ER}(n,h_n)$ and a binomial random intersection graph $G_s(n,t_n,P_n)$ obey

$$G_{ER}(n,h_n) \preceq_{1-o(1)} G_s(n,t_n,P_n). \tag{40}$$

F. Couplings between a binomial random s-intersection graph an a uniform random s-intersection graph

Lemma 12 ([3, Lemma 4])**.** *If $t_n P_n = \omega(\ln n)$, and for all n sufficiently large,*

$$K_n^- \leq t_n P_n - \sqrt{3(t_n P_n + \ln n)\ln n},$$
$$K_n^+ \geq t_n P_n + \sqrt{3(t_n P_n + \ln n)\ln n},$$

then

$$\begin{aligned} H_s(n,K_n^-,P_n) &\preceq_{1-o(1)} G_s(n,t_n,P_n) \\ &\preceq_{1-o(1)} H_s(n,K_n^+,P_n). \end{aligned}$$

Lemma 13. *If $K_n = \omega(\ln n)$, then with $t_n^- = \frac{K_n}{P_n}\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right)$ and $t_n^+ = \frac{K_n}{P_n}\left(1 + \sqrt{\frac{3\ln n}{K_n}}\right)$, it holds that*

$$\begin{aligned} G_s(n,t_n^-,P_n) &\preceq_{1-o(1)} H_s(n,K_n,P_n) \\ &\preceq_{1-o(1)} G_s(n,t_n^+,P_n). \end{aligned}$$

G. The Proof of Lemma 13

We use Lemma 12 to prove Lemma 13. From conditions $K_n = \omega(\ln n)$ and $t_n^{\pm} = \frac{K_n}{P_n}\left(1 \pm \sqrt{\frac{3\ln n}{K_n}}\right)$, it holds that $t_n^{\pm} P_n = \omega(\ln n)$. For all n sufficiently large, we obtain

$$\begin{aligned} &K_n - \left[t_n^- P_n + \sqrt{3(t_n^- P_n + \ln n)\ln n}\right] \\ &= K_n \sqrt{\frac{3\ln n}{K_n}} - \sqrt{3\left[K_n\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right) + \ln n\right]\ln n} \\ &= \sqrt{3K_n \ln n} - \sqrt{3\left[K_n + \sqrt{\ln n}\left(\sqrt{\ln n} - \sqrt{3K_n}\right)\right]\ln n} \\ &\geq \sqrt{3K_n \ln n} - \sqrt{3K_n \ln n} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &K_n - \left[t_n^+ P_n - \sqrt{3(t_n^+ P_n + \ln n)\ln n}\right] \\ &= -K_n \sqrt{\frac{3\ln n}{K_n}} + \sqrt{3\left[K_n\left(1 + \sqrt{\frac{3\ln n}{K_n}}\right) + \ln n\right]\ln n} \\ &\leq -\sqrt{3K_n \ln n} + \sqrt{3K_n \ln n} \\ &= 0. \end{aligned}$$

Then by Lemma 12, we have $G_s(n,t_n^-,P_n) \preceq_{1-o(1)} H_s(n,K_n,P_n)$ and $H_s(n,K_n,P_n) \preceq_{1-o(1)} G_s(n,t_n^+,P_n)$, so Lemma 13 is now established.